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Spectral Methods for Linear Inverse Problems with Unbounded Operators*

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Spectral theory for bounded linear operators is used to develop a general class of approximation methods for the Moore–Penrose generalized inverse of a closed, densely defined linear operator. Issues of convergence and stability are addressed and the methods are modified to provide a stable class of methods for evaluation of unbounded linear operators. © 1992 Academic Press, Inc.

1. INTRODUCTION

A great many linear inverse problems of mathematical physics may be framed in an abstract setting as linear operator equations of the form

$$Ax = f \tag{1.1}$$

which implicitly define the solution x of the given problem. The desired solution x is often given in terms of the Moore–Penrose generalized inverse $A^\dagger$ in the form $x = A^\dagger f$. In all interesting cases the generalized inverse is an unbounded operator and the challenge is then to provide approximations to the unknown solution $A^\dagger f$ that are stable with respect to perturbations in the data f. This is the subject of regularization theory and there is a large literature on regularization for the case in which A is a compact, or bounded linear operator (see, e.g., [17, 4, 7, 1, 11]). However, when A is a closed, densely defined linear operator it appears that relatively few special results are known (see, e.g., [9, 13, 15]). For closed, densely defined linear operators a general theory of regularization would subsume the theory for bounded linear operators (e.g., [3, 4]) and would also apply not only to

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linear integral equations of the first kind, but also to certain intregro-differential equations, problems involving best approximate solution of two-point boundary value problems [8], and Cauchy problems for elliptic partial differential equations [10]. One of the aims of this paper is to provide an outline of a general theory of regularization for closed, densely defined, possibly unbounded, linear operators.

Some linear inverse problems can be explicitly inverted to give a solution of the form

$$x = Lf, \tag{1.2}$$

where L is a known, but unbounded, linear operator. The solution x is then unstable with respect to small perturbations in f. Basic examples of this type would include, for instance, numerical differentiation, the explicit solution of Abel equations, and the estimation of normal derivatives from Dirichlet data for elliptic boundary value problems. For the explicit equation (1.2) the challenge is to provide stable methods for computing Lf when f is only approximately known and the approximate data perhaps do not belong to the domain of L.

Our goal is to develop a general framework, based on spectral theory for bounded linear operators in Hilbert space, for stable approximate solution of abstract inverse problems, posed in either implicit or explicit form, involving closed, densely defined linear operators which may be unbounded. It is our intension to frame the broad outlines of the theory, but we will not investigate all details. In particular, questions of optimal a posteriori parameter choice and finite dimensional approximations, as worked out in [2] and [5], respectively, for bounded operators will not be addressed.

2. The Implicit Problem

The equation

$$Ax = f, \tag{2.1}$$

where A is a linear operator from a Hilbert space H_1 into a Hilbert space H_2 (we denote the inner product, norm, and identity operator in any Hilbert space by $(\cdot, \cdot)$, $\|\cdot\|$, and I, respectively) is guaranteed to have a unique solution only when f lies in $R(A)$, the range of A, and the nullspace of A, $N(A)$, is trivial. However, there is a well-known formalism for assigning a unique pseudo-solution to (2.1) in the case when $N(A)$ is nontrivial and f lies in a certain superspace of $R(A)$ which is dense in H_2.

Specifically, if A is closed and defined on a dense subspace $\mathscr{D}(A) \subset H_1$, then a unique pseudo-solution can be defined for each f in the dense subspace $R(A) + R(A)^{\perp}$ of H_2 by $x = A^{\dagger}f$, where $x \in \mathscr{D}(A) \cap N(A)^{\perp}$ satisfies

$$Ax = Qf \tag{2.2}$$

and Q is the orthogonal projector of H_2 onto $\overline{R(A)}$, the closure of $R(A)$. The operator $A^{\dagger}$ so defined on $\mathscr{D}(A^{\dagger}) = R(A) + R(A)^{\perp}$ is called the Moore–Penrose generalized inverse of A and provides a unique generalized solution of (2.1) (the least squares solution of minimum norm) in cases in which classical solutions might not exist, or existing, might not be unique.

The operator $A^{\dagger}$ is a closed linear operator which is bounded if and only if the range of A is closed (see [6, 9] for basic information on the generalized inverse of a closed linear operator). For purposes of stable approximation of generalized solutions in the presence of errors in f it is therefore important to approximate $A^{\dagger}$ by bounded linear operators. In [3, 4] a general approach to such approximations is developed for bounded linear operators A. Lardy [9] has given a series representation for $A^{\dagger}$ when A is a closed densely defined linear operator. In this section a general stable method for approximating $A^{\dagger}$ when A is densely defined and closed, but possibly unbounded, is provided. The development is a generalization of the corresponding theory for bounded operators. The general results are illustrated for certain specific continuous and iterative approximation methods.

Our representation of $A^{\dagger}$ as a limit of bounded linear operators and the associated theory of regularization parallels the development in [3, 4] and may be motivated by some purely formal considerations. We begin with the well-known fact that

$$A^{\dagger}Ax = Px \qquad \text{for} \quad x \in \mathscr{D}(A),$$

where P is the projector of H_1 onto $\overline{R(A^*)}$. Therefore

$$A^{\dagger}AA^*y = A^*y \qquad \text{for} \quad y \in \mathscr{D}(AA^*).$$

That is, $A^{\dagger}AA^*$ and A^* coincide on a dense subspace of H_1. We see that if these operators were everywhere defined and if the inverses involved existed, then *formally* we would have

$$A^{\dagger} = A^*(AA^*)^{-1} = A^*(I + AA^*)^{-1}(I - (I + AA^*)^{-1})^{-1}.$$

It is remarkable that, by a theorem of von Neumann, the operators $A^*(I + AA^*)^{-1}$ and $(I + AA^*)^{-1}$ are *everywhere defined and bounded.* Moreover, $(I + AA^*)^{-1}$ is a bounded self-adjoint operator with spectrum contained in $[0, 1]$. However, if $R(A)$ is not closed, then the operator $I-(I+AA^*)^{-1}$ does not have a bounded inverse and hence we are led to replace the right hand side in the formal representation of $A^\dagger$ given above by

$$A^*(I+AA^*)^{-1}\mathcal{S}_\alpha((I+AA^*)^{-1}),$$

where $\mathcal{S}_\alpha(t)$ $(\alpha>0)$ is a family of continuous real-valued functions on $[0, 1]$ that approximates $1/(1-t)$ in an appropriate sense.

Having laid the intuitive basis of the general method of approximating $A^\dagger$ by bounded linear operators, we now proceed to a systematic development of approximation and regularity results. We begin by stating a fundamental theorem of von Neumann (see [14, Chap. 8; 9]). It will be convenient here and in the sequel to use the notation $\hat{A}:=(I+AA^*)^{-1}$ and $\check{A}:=(I+A^*A)^{-1}$.

PROPOSITION 2.1. *If A is a closed densely defined linear operator, then $\hat{A}:=(I+AA^*)^{-1}$, $A^*\hat{A}$, $\check{A}:=(I+A^*A)^{-1}$, and $A\check{A}$ are bounded (everywhere defined) linear operators. Moreover, $\hat{A}$ and $\check{A}$ are self-adjoint and their spectra lie in* $[0, 1]$.

We now need a technical lemma.

LEMMA 2.2. *If $g\in C[0, 1]$, then $A^*g(\hat{A})y=g(\check{A})A^*y$ for all $y\in\mathcal{D}(A^*)$* and *$Ag(\check{A})x=g(\hat{A})Ax$ for all $x\in\mathcal{D}(A)$.*

Proof. Let $z=A^*\hat{A}y$, where $y\in\mathcal{D}(A^*)$. Since $\hat{A}y\in\mathcal{D}(AA^*)$, we have $z\in\mathcal{D}(A)$. Also, $Az=-\hat{A}y+y\in\mathcal{D}(A^*)$, hence $z\in\mathcal{D}(A^*A)$ and

$$(I+A^*A)z=(I+A^*A)A^*\hat{A}y=A^*(I+AA^*)\hat{A}y=A^*y.$$

Therefore,

$$A^*\hat{A}y=z=(I+A^*A)^{-1}A^*y=\check{A}A^*y.$$

From this it follows that

$$A^*p(\hat{A})y=p(\check{A})A^*y, \qquad \text{for} \quad y\in\mathcal{D}(A^*),$$

where p is any polynomial. Now let $\{p_n\}$ be a sequence of polynomials converging uniformly to g on $[0, 1]$. Then for any $y\in\mathcal{D}(A^*)$ and $x\in\mathcal{D}(A)$,

$$\begin{aligned}(A^*g(\hat{A})y, x)&=(g(\hat{A})y, Ax)=\lim_n(p_n(\hat{A})y, Ax)\\&=\lim_n(A^*p_n(\hat{A})y, x)=\lim_n(p_n(\check{A})A^*y, x)=(g(\check{A})A^*y, x)\end{aligned}$$

and hence $A^*g(\hat{A})\,y = g(\check{A})\,A^*y$. The other equality is established in the same way. ▮

Suppose now that $\{\mathscr{S}_\alpha\}_{\alpha>0}$ is a family of continuous real-valued functions on $[0, 1]$ satisfying

$$\lim_{\alpha\to 0}(1-t)\,\mathscr{S}_\alpha(t) = 1 \qquad \text{for each} \quad t\in[0,1) \tag{2.3}$$

$$|(1-t)\,\mathscr{S}_\alpha(t)| \text{ is uniformly bounded} \tag{2.4}$$

(we also occasionally use an index $\beta\to\infty$, or even a discrete index $n\to\infty$, without special notice). Approximations x_α to $A^\dagger f$ will be formed in the following way:

$$x_\alpha = A^*\hat{A}\mathscr{S}_\alpha(\hat{A})f. \tag{2.5}$$

A notable feature of these approximations is their stability; that is, while $A^\dagger f$ depends discontinuously of f (since $A^\dagger$ is unbounded), the operators $A^*\hat{A}$ and $\mathscr{S}_\alpha(\hat{A})$ are both bounded and hence x_α, for each fixed $\alpha>0$, depends continuously on f.

Some simple examples are in order. The method resulting in Tikhonov regularization [17] is given by

$$\mathscr{S}_\alpha(t) = \frac{1}{1+(\alpha-1)t}. \tag{2.6}$$

The iteratively defined sequence

$$\mathscr{S}_0(t) = 0, \qquad \mathscr{S}_{n+1}(t) = 1 + t\mathscr{S}_n(t) \tag{2.7}$$

results in Lardy's method [9], while for $\beta\to\infty$

$$\mathscr{S}_\beta(t) = \frac{1}{t}\int_0^\beta e^{-((1-t)/t)u}\,du, \qquad \mathscr{S}_\beta(0) = 1 \tag{2.8}$$

gives a representation for $A^\dagger$ which was studied by Showalter [16] for bounded linear operators. Each of these examples has a variational interpretation in terms of minimization of a least squares functional. In general, we have

$$x_\alpha = A^*\hat{A}\mathscr{S}_\alpha(\hat{A})f \in \mathscr{D}(A), \qquad \text{since} \quad R(\hat{A})\subseteq\mathscr{D}(AA^*).$$

Also,

$$Ax_\alpha = (I-\hat{A})\,\mathscr{S}_\alpha(\hat{A})f.$$

For (2.6) we then have, for each $v \in \mathscr{D}(A)$,

$$(Ax_\alpha, Av) = ((I - \hat{A})(I + (\alpha - 1)\hat{A})^{-1} f, Av)$$
$$= (f, Av) - \alpha(A^* \hat{A} \mathscr{S}_\alpha(\hat{A}) f, v),$$

that is,

$$(Ax_\alpha, Av) + \alpha(x_\alpha, v) = (f, Av), \qquad \text{for all} \quad v \in \mathscr{D}(A).$$

In this form (2.6) gives what Lattès and Lions [10] call the method of quasi-reversibility for the closed unbounded operator A. On the other hand, this equation is the Euler equation for the minimization of the penalized least squares functional

$$\|Ax - f\|^2 + \alpha \|x\|^2$$

over $\mathscr{D}(A)$ and hence (2.6) is in fact Tikhonov's method [17]. In a similar way, the method (2.7) of Lardy is seen to satisfy $x_0 = 0$ and

$$(Ax_{n+1}, Av) + (x_{n+1} - x_n, v) = (f, Av), \qquad \text{for all} \quad v \in \mathscr{D}(A)$$

which is the Euler equation for the functional

$$\|Ax - f\|^2 + \|x - x_n\|^2, \qquad x \in \mathscr{D}(A).$$

Therefore Lardy's method may be interpreted as an iterative least squares method in which the penalty term $\|x - x_n\|^2$ has a stability influence on the new approximation x_{n+1}. Finally, for the method (2.8) it is easy to check that

$$\frac{dx_\beta}{d\beta} = -(A^*Ax_\beta - A^*f), \qquad \text{for} \quad f \in \mathscr{D}(A^*).$$

However, $A^*Ax - A^*f$ is the gradient of the least squares functional $\frac{1}{2}\|Ax - f\|^2$ and hence (2.8) may be interpreted as a method of continuously following the trajectory of steepest descent for the least squares functional.

We now present the basic convergence theorem.

THEOREM 2.3. *If $f \in \mathscr{D}(A^\dagger)$, then $x_\alpha \to A^\dagger f$ in graph norm as $\alpha \to 0$.*

Proof. Note that

$$f = Qf + (I - Q)f = AA^\dagger f + (I - Q)f \in R(A) + N(A^*).$$

Therefore, by Lemma 2.2,

$$x_\alpha = A^*\hat{A}\mathscr{S}_\alpha(\hat{A})f = A^*\hat{A}\mathscr{S}_\alpha(\hat{A})\,AA^\dagger f + \check{A}\mathscr{S}_\alpha(\check{A})\,A^*(I-A)f$$
$$= A^*A\check{A}\mathscr{S}_\alpha(\check{A})\,A^\dagger f = (I-\check{A})\,\mathscr{S}_\alpha(\check{A})\,A^\dagger f. \tag{2.9}$$

We then find from (2.3), (2.4), and the spectral representation of the bounded self-adjoint operator $\check{A}$ that

$$x_\alpha - A^\dagger f = (I-(I-\check{A})\,\mathscr{S}_\alpha(\check{A}))\,A^\dagger f \to P_{N(\check{A})}A^\dagger f = 0, \qquad \text{as} \quad \alpha \to 0,$$

where $P_{N(\check{A})}$ is the projector of H_1 onto $N(\check{A}) = \{0\}$. Also, since $\hat{A}$ maps H_2 into $\mathscr{D}(AA^*)$, it follows from (2.5), (2.9), and Lemma 2.2 that $x_\alpha \in \mathscr{D}(A)$ and

$$Ax_\alpha = (I-\hat{A})\,\mathscr{S}_\alpha(\hat{A})\,AA^\dagger f = (I-\hat{A})\,\mathscr{S}_\alpha(\hat{A})\,Qf$$

and hence

$$Ax_\alpha - AA^\dagger f = (I-(I-\hat{A})\,\mathscr{S}_\alpha(\hat{A}))\,Qf \to P_{N(\hat{A})}\,Qf = 0, \qquad \text{as} \quad \alpha \to 0.$$

Therefore $\{x_\alpha\}$ converges in graph norm to $A^\dagger f$. ▮

Under appropriate conditions convergence rates may be obtained. For example,

THEOREM 2.4. *If* $f \in \mathscr{D}(A^\dagger)$ *and* $A^\dagger f \in R(A^*A)$, *then* $\|x_\alpha - A^\dagger f\| = O(\omega(\alpha))$, *where* $\omega(\alpha) = \max_{\tau \in [0,1]} |\tau(1-\tau\mathscr{S}_\alpha(1-\tau))|$.

Proof. Suppose $A^\dagger f = A^*Aw$. Let $z = w + A^\dagger f$; then

$$(I-\check{A})z = w + A^\dagger f - \check{A}w - \check{A}((I+A^*A)w - w) = A^\dagger f.$$

Therefore by (2.9),

$$x_\alpha - A^\dagger f = (I-\check{A})(I-(I-\check{A})\,\mathscr{S}_\alpha(\check{A}))z$$

and the result follows. ▮

For the methods (2.6), (2.7), and (2.8) one has $\omega(\alpha) = \alpha$, $\omega(n) = O(1/n)$, and $\omega(\beta) = O(1/\beta)$, respectively. Rates of the above type can also be obtained for approximation of certain functionals of $A^\dagger f$ by transferring the regularity condition from the solution to the functional. In fact, if one approximates the functional $(A^\dagger f, \theta)$ by (x_α, θ), then, if $\theta = A^*Aw$, one obtains, as in the proof of Theorem 2.4, $\theta = (I-\check{A})w$ and hence

$$|(A^\dagger f, \theta) - (x_\alpha, \theta)| = \|((I-\check{A})[x_\alpha - A^\dagger f], w)| = O(\omega(\alpha)).$$

If stronger assumptions are made on A, then a correspondingly stronger convergence result may be achieved. If we assume that $R(A)$ is closed, then it is well-known that $A^\dagger$ is a bounded operator [6, 9] and in this case convergence in operator norm is possible.

THEOREM 2.5. *If $R(A)$ is closed and the convergence in* (2.3) *is uniform on closed subintervals of* $[0, 1)$, *then*

$$\|A^\dagger - A^*\hat{A}\mathscr{S}_\alpha(\hat{A})\| \to 0 \qquad \text{as} \quad \alpha \to 0.$$

Proof. Since $R(A)$ is closed, $A^\dagger\colon H_2 \to N(A)^\perp$ is everywhere defined and bounded. Also, by the closed graph theorem, there is a number $m > 0$ such that

$$\|Ax\| \geqslant m\,\|x\| \qquad \text{for} \quad x \in N(A)^\perp.$$

From this it follows that, considered as an operator on $N(A)^\perp$, we have $\|\check{A}\| < 1$. Indeed, if $x \in N(A)^\perp$, then $x = (I + A^*A)w$ for some $w \in \mathscr{D}(A^*A) \cap N(A)^\perp$. One then has

$$\|x\|\;\|w\| \geqslant (x, w) = \|w\|^2 + \|Aw\|^2 \geqslant (1 + m^2)\;\|w\|^2;$$

therefore

$$\|\check{A}x\| = \|w\| \leqslant \frac{1}{1+m^2}\,\|x\|.$$

But, as in the proof of Theorem 2.3, we have

$$A^\dagger - A^*\hat{A}\mathscr{S}_\alpha(\hat{A}) = (I - (I - \check{A})\,\mathscr{S}_\alpha(\hat{A}))A^\dagger \tag{2.10}$$

and the result follows since $R(A^\dagger) \subseteq N(A)^\perp$. ∎

Uniform convergence rates in Theorem 2.5 may be provided in terms of the function

$$\Omega(\alpha) = \max_{t \in [0,\mu]} |1 - (1-t)\,\mathscr{S}_\alpha(t)|, \qquad \text{where} \quad \mu = \frac{1}{1+m^2}.$$

In fact, from (2.10),

$$\|A^\dagger - A^*\hat{A}\mathscr{S}_\alpha(\hat{A})\| \leqslant \Omega(\alpha)\;\|A^\dagger\|.$$

For example, in the methods (2.6), (2.7), and (2.8), one has $\Omega(\alpha) = O(\alpha)$, $\Omega(n) = \mu^{n+1}$, and $\Omega(\beta) = \exp(-(1-\mu)\,\beta/\mu)$, respectively.

In contrast to the strong convergence results given above, if less is assumed of f, namely that $f \notin \mathscr{D}(A^\dagger)$, then not even weak convergence can be expected.

THEOREM 2.6. *If* $\alpha_n \to 0$ *and* $x_{\alpha_n} \xrightarrow{w} z$ *(weak convergence), then* $f \in \mathscr{D}(A^\dagger)$ *and* $A^\dagger f = z$.

Proof. As in the proof of Theorem 2.3, we have

$$Ax_{\alpha_n} = (I - \hat{A})\,\mathscr{S}_{\alpha_n}(\hat{A}) f \to Qf.$$

But the graph of A is closed and convex, and therefore weakly closed. It follows that $Az = Qf$, that is,

$$f \in R(A) + R(A)^\perp = \mathscr{D}(A^\dagger).$$

Also, $x_{\alpha_n} \in R(A^*) \subseteq N(A)^\perp$ and therefore, since $N(A)^\perp$ is weakly closed, we find that $z \in N(A)^\perp$, that is, $z = A^\dagger f$. ∎

As an immediate corollary we obtain the following nonconvergence result which generalizes a theorem of Maslov [12].

COROLLARY 2.7. *If* $f \notin \mathscr{D}(A^\dagger)$, *then* $\|x_\alpha\| \to \infty$ *as* $\alpha \to 0$.

We now investigate the effect of data perturbations on the approximations $\{x_\alpha\}$. Suppose that only approximate data $f^\delta \in H_2$ are available, where $\delta \geqslant \|f - f^\delta\|$ is a known estimate for the quality of these data. An approximation to $A^\dagger f$ using the available data is then given by

$$x_\alpha^\delta = A^* \hat{A} \mathscr{S}_\alpha(\hat{A}) f^\delta. \tag{2.11}$$

Since the approximate data are described by an arbitrary function f^δ satisfying $\|f - f^\delta\| \leqslant \delta$, one can expect in general that $f^\delta \notin \mathscr{D}(A^\dagger)$ and hence, by Corollary 2.7, $\|x_\alpha^\delta\| \to \infty$ as $\alpha \to 0$ for fixed $\delta > 0$. This raises the question of Tikhonov regularity of (2.11), that is, the possibility of a choice $\alpha = \alpha(\delta) \to 0$ as $\delta \to 0$ in such a way that $x_{\alpha(\delta)}^\delta \to A^\dagger f$ as $\delta \to 0$. Such a regularity condition can be given in terms of the quantity

$$r(\alpha) = \max_{t \in [0,1]} |t \mathscr{S}_\alpha(t)|. \tag{2.12}$$

THEOREM 2.8. *Suppose* $f \in \mathscr{D}(A^\dagger)$ *and* $\alpha = \alpha(\delta) \to 0$ *as* $\delta \to 0$. *If* $\delta \sqrt{r(\alpha(\delta))} \to 0$, *then* $x_{\alpha(\delta)}^\delta \to A^\dagger f$ *as* $\delta \to 0$.

Proof. Since $x_\alpha \to A^\dagger f$ as $\alpha \to 0$, it suffices to consider the quantity $\|x_{\alpha(\delta)}^\delta - x_{\alpha(\delta)}\|^2$. For this we have, since $R(\hat{A}) \subseteq \mathscr{D}(AA^*)$,

$$\begin{aligned}\|x_{\alpha(\delta)}^\delta - x_{\alpha(\delta)}\|^2 &= (A^*\hat{A}\mathscr{S}_\alpha(\hat{A})(f^\delta - f), A^*\hat{A}\mathscr{S}_\alpha(\hat{A})(f^\delta - f)) \\ &= (\hat{A}\mathscr{S}_\alpha(\hat{A})(f^\delta - f), (I - \hat{A})\,\mathscr{S}_\alpha(\hat{A})(f^\delta - f)).\end{aligned}$$

However, $\|(I-\hat{A})\,\mathscr{S}_\alpha(\hat{A})\|$ is bounded by (2.3) and it follows that $\|x^\delta_{\alpha(\delta)} - x_{\alpha(\delta)}\|^2 \leqslant \delta^2 r(\alpha(\delta))$. ∎

3. The Explicit Problem

In the preceding section a general approach to forming stable approximations to an unknown unbounded generalized inverse operator was developed. It is often the case that, as opposed to computing an unknown generalized inverse, one wishes only to compute stable approximations to the values of a known unbounded operator L in order to get stable approximations to the explicit solution of an inverse problem as given by (1.2). A simple case in point is the problem of approximate differentiation in which an unbounded operator (the derivative) is to be applied to a function that is known only approximately and the approximate function (the data) might in fact be nondifferentiable. Moreover, even if the approximate data are differentiable, small errors in the data might be magnified to unacceptable levels by the action of the unbounded derivative operator. These considerations lead us to develop in this section a general stable approximation scheme for the evaluation of unbounded linear operators. Our presentation will be relatively brief as the ideas and techniques are similar to those of the previous section. For a more extensive computational treatment of a particular method based on Tikhonov regularization for evaluation of unbounded operators see Morozov [13, Chap. 4].

Consider a closed, densely defined unbounded linear operator $L: \mathscr{D}(L) \subseteq H_1 \to H_2$ from a Hilbert space H_1 into a Hilbert space H_2. Given $x \in \mathscr{D}(L)$ we wish to form stable (i.e., continuous in x) approximations to the element

$$y = Lx.$$

Suppose that $\{T_\alpha\}_{\alpha>0}$ is a family of continuous real-valued functions on $[0, 1]$ satisfying

$$T_\alpha(t) \to 1/t \text{ as } \alpha \to 0 \text{ for each } t \in (0, 1]$$

and (3.1)

$$|tT_\alpha(t)| \text{ is uniformly bounded.}$$

Such a family may be formed by taking $T_\alpha(t) = \mathscr{S}_\alpha(1-t)$, where $\mathscr{S}_\alpha(t)$ satisfies (2.3) and (2.4). Then, for each $\alpha > 0$, the elements

$$y_\alpha = L\check{L}T_\alpha(\check{L})x \tag{3.2}$$

are defined for each $x \in H_1$ and continuous in x since $L\check{L}$ and $T_\alpha(\check{L})$ are bounded linear operators. We will show $y_\alpha \to Lx$ as $\alpha \to 0$ and that this approximation scheme is regular with respect to perturbations in x under appropriate circumstances.

Two particularly simple examples of methods of this type are methods based on Tikhonov regularization and Lardy's method given by

$$T_\alpha(t) = \frac{1}{\alpha + (1-\alpha)t} \tag{3.3}$$

and

$$T_0(t) = 0, \qquad T_{n+1}(t) = 1 + (1-t)\, T_n(t), \tag{3.4}$$

respectively. The choice (3.3) gives rise to the method

$$y_\alpha = L(\alpha L^* L + I)^{-1} x \tag{3.5}$$

which has been studied by Morozov [13] and the choice (3.4) gives the simple iterative method

$$y_1 = L\check{L}x, \qquad y_{n+1} = \check{L}Lx + (I - \check{L})\, y_n. \tag{3.6}$$

THEOREM 3.1. *If $x \in \mathscr{D}(L)$, then $y_\alpha \to Lx$ as $\alpha \to 0$.*

Proof. We have

$$Lx - y_\alpha = (L - L\check{L}T_\alpha(\check{L}))x = (I - \check{L}T_\alpha(\check{L}))\, Lx;$$

however, $I - \check{L}T_\alpha(\check{L})$ converges strongly to zero by (3.1). ∎

THEOREM 3.2. *If $y_{\alpha_n} \xrightarrow{w} y$ for some sequence $\alpha_n \to 0$, then $x \in \mathscr{D}(L)$ and $Lx = y$.*

Proof. Note that $y_{\alpha_n} = L\check{L}T_{\alpha_n}(\check{L})x$ and $\check{L}T_{\alpha_n}(\check{L})x \to x$ as $\alpha_n \to 0$. Therefore, since the graph of L is weakly closed, we have $x \in \mathscr{D}(L)$ and $Lx = y$. ∎

COROLLARY 3.3. *If $x \notin \mathscr{D}(L)$, then $\|y_\alpha\| \to \infty$ as $\alpha \to 0$.*

Under additional assumptions convergence rates may be provided.

THEOREM 3.4. *If $x \in \mathscr{D}(L) \cap R(L^*)$, then $\|y_\alpha - Lx\| = O(\omega(\alpha))$, where $\omega(\alpha) = \max_{t \in [0,1]} |(1 - tT_\alpha(t))t|$.*

Proof. If $x \in \mathscr{D}(L)$ and $x = L^*z$, then

$$y_\alpha - Lx = L\check{L}T_\alpha(\check{L})x - Lx = (I - \hat{L}T_\alpha(\hat{L}))Lx = (I - \hat{L}T_\alpha(\hat{L}))\,\hat{L}z$$

and the result follows immediately. ▮

Note that $\omega(\alpha)$ is the same function as in Theorem 2.4 (with $T_\alpha(t) = \mathscr{S}(1-\tau)$). In particular for the method (3.5) we see that the rate $O(\alpha)$ is attainable. The next theorem shows that this rate is essentially the best possible.

THEOREM 3.5. *Suppose* $x \in \mathscr{D}(L)$ *and* $\|y_\alpha - Lx\| = o(\alpha)$, *where* y_α *is given by* (3.5), *then* $x \in N(L)$.

Proof. Let

$$\begin{aligned} e_\alpha = y_\alpha - Lx &= L\check{L}[\alpha I + (1-\alpha)\check{L}]^{-1}x - Lx \\ &= \{[\alpha I + (1-\alpha)\hat{L}]^{-1}\hat{L} - I\}\,Lx. \end{aligned}$$

Then

$$[\alpha I + (1-\alpha)\hat{L}]\,e_\alpha = \alpha(\hat{L} - I)Lx.$$

Since $\alpha I + (1-\alpha)\hat{L}$ is bounded and $\|e_\alpha\| = o(\alpha)$, we find that

$$(\hat{L} - I)Lx = 0 \qquad \text{or} \qquad \hat{L}Lx = Lx.$$

Therefore, $Lx \in R(\hat{L}) \subseteq \mathscr{D}(LL^*)$ and

$$Lx = (I + LL^*)Lx,$$

that is, $LL^*Lx = 0$. Hence

$$0 = (LL^*Lx, Lx) = \|L^*Lx\|^2,$$

that is, $Lx \in R(L) \cap N(L^*) = \{0\}$. ▮

We now investigate the influence of errors in the data. Suppose that the available data are given by a function x^δ (which need not lie in $\mathscr{D}(L)$)), where $\|x^\delta - x\| \leqslant \delta$. We may form the approximations

$$y_\alpha^\delta = L\check{L}T_\alpha(\check{L})x^\delta.$$

Since in general $x^\delta \notin \mathscr{D}(L)$, we may expect that $\|y_\alpha^\delta\| \to \infty$ as $\alpha \to 0$ for fixed δ. For regularity we would seek a choice $\alpha = \alpha(\delta) \to 0$ such that $y^\delta_{\alpha(\delta)} \to Lx$ as $\delta \to 0$.

THEOREM 3.6. *If* $\delta\sqrt{r(\alpha(\delta))} \to 0$ *as* $\delta \to 0$, *where* $r(\alpha)$ *is given by* (2.12) (*with* $T_\alpha(t) = \mathcal{S}_\alpha(1-t)$), *then* $y^\delta_{\alpha(\delta)} \to Lx$ *as* $\delta \to 0$.

Proof. As in the proof of Theorem 2.8, we find that

$$\|y^\delta_{\alpha(\delta)} - y_{\alpha(\delta)}\|^2 = |(\check{L}T_{\alpha(\delta)}(\check{L})(x^\delta - x), (I-\check{L})\, T_{\alpha(\delta)}(\check{L})(x^\delta - x))|$$
$$\leqslant \delta^2 r(\alpha(\delta))\, \|\check{L}T_\alpha(\check{L})\|,$$

and the result follows. ∎

REFERENCES

1. J. BAUMEISTER, "Stable Solution of Inverse Problems," Vieweg, Braunschweig, 1987.
2. H. W. ENGL AND H. GFRERRER, A posteriori parameter choice for general regularization methods for solving linear ill-posed problems, *Appl. Numer. Math.* **4** (1988), 395–417.
3. C. W. GROETSCH, "Generalized Inverses of Linear Operators: Representation and Approximation," Dekker, New York, 1977.
4. C. W. GROETSCH, "The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind," Pitman, Boston, 1984.
5. C. W. GROETSCH AND A. NEUBAUER, Regularization of ill-posed problems: Optimal parameter choice in finite dimensions, *J. Approx. Theory* **58** (1989), 184–200.
6. M. R. HESTENES, Relative self-adjoint operators in Hilbert space, *Pacific J. Math.* **11** (1961), 1315–1357.
7. B. HOFMANN, "Regularization of Applied Inverse and Ill-posed Problems," Teubner, Leipzig, 1986.
8. W. F. LANGFORD, A shooting algorithm for the best least squares solution of two-point boundary value problems, *SIAM J. Numer. Anal.* **14** (1977), 527–542.
9. L. J. LARDY, A series representation of the generalized inverse of a closed linear operator, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **58** (1975), 152–157.
10. R. LATTÈS AND J.-L. LIONS, "The Method of Quasi-reversibility," American Elsevier, New York, 1969.
11. A. K. LOUIS, "Inverse und schlecht gestellte Probleme," Teubner, Stuttgart, 1989.
12. V. P. MASLOV, The existence of a solution to an ill-posed problem is equivalent to the convergence of a regularization process, *Uspekhi Mat. Nauk* **23**, No. 3 (1968), 183–184. [In Russian]
13. V. A. MOROZOV, "Methods for Solving Incorrectly Posed Problems," Springer-Verlag, New York, 1984.
14. F. RIESZ AND B. SZ.-NAGY, "Functional Analysis," Ungar, New York, 1955.
15. E. SCHOCK AND V. Q. PHONG, Regularization of ill-posed problems involving unbounded operators in Banach spaces, preprint.
16. D. SHOWALTER, Representation and computation of the pseudoinverse, *Proc. Amer. Math. Soc.* **18** (1967), 584–586.
17. A. N. TIKHONOV AND V. Y. ARSENIN, "Solutions of Ill-posed Problems," Wiley, New York, 1977.